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Obstruction and Some Approximate Controllability Results for the Burgers Equation and Related Problems

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1 INTRODUCTION

It is well accepted that the Burgers equation provides a realistic simplification of the Navier-Stokes system in Fluid Mechanics. One of the most important common features among both equations is the presence of a quadratic nonlinear term and a linear diffusion operator. Some *optimal control* results associated to the Burgers equation have been obtained in the literature (see *e.g.*, Glowinski and Lions (1994) and its references). This work concerns with other type of control problem associated to the Burgers equation: the approximate controllability for a final observation. To fix ideas, given some positive numbers T, L and ν we consider the following boundary control problem

$$\left.\begin{array}{ll} y_t + yy_x - \nu y_{xx} = 0 & \text{in } (0,T)\times(0,L) \\ y(t,0)=0, \quad y(t,L)=u(t), & t\in(0,T) \\ y(0,x)=y_0(x), & x\in(0,L) \end{array}\right\} \qquad (1)$$

where the control $u(t)$ is assumed to be a function on $(0,T)$ and the initial datum is given. For the sake of simplicity in the exposition we shall assume that $y_0 \in L^\infty(0,L)$. Other controllability problems associated to the Burgers equation are also considered (see Remark 1.5). We recall that the approximate controllability property, with final observation in a Banach space of states, $\mathcal{X}$, of functions defined on $(0,L)$ (*e.g.* $\mathcal{X} = \mathcal{C}([0,L])$ or $\mathcal{X} = L^2(0,L)$) can be stated in the following terms: given a desired state $y_d \in \mathcal{X}$ and $\varepsilon > 0$ find a control $u_\varepsilon(t)$ such that the solution $y(t,\cdot;u_\varepsilon)$ of (1) corresponding to $u = u_\varepsilon$ satisfies that

$$\|y(T,\cdot:u_\varepsilon) - y_d\|_{\mathcal{X}} \leq \varepsilon. \tag{2}$$

In Section 2 we shall prove that this property can not holds under this general statement. To do that, we shall show that an *Obstruction Phenomenon* arises due to the presence of the superlinear term $(y^2)_x/2$ at the equation. This Obstruction Phenomenon was already exhibited for the case of semilinear parabolic problems in a series of works: Henry (1978), Díaz (1991a), (1991b), (1993a), (1994a) and (1994b), Díaz and Ramos (1993), (1994), Bernis, Díaz and Ramos (1995).The results here presented complete and generalize previous considerations on the Burgers equation made in Díaz (1991a) and (1991b). As an application we give a necessary condition for the approximate controllability of the Navier-Stokes system on a rectangle $\Omega = (0,L_1)\times(0,L_2)$. For $\Gamma = \{L_1\}\times[0,L_2]$, we consider the boundary control problem

$$\left.\begin{array}{ll} \mathbf{y}_t + (\mathbf{y}\cdot\nabla)\mathbf{y} - \nu\Delta\mathbf{y} = -\nabla p & \text{in } (0,T)\times\Omega \\ \operatorname{div}\mathbf{y} = 0 & \text{in } (0,T)\times\Omega \\ \mathbf{y} = 0 & \text{on } (0,T)\times(\partial\Omega\setminus\Gamma) \\ \mathbf{y} = \mathbf{u} & \text{on } (0,T)\times\Gamma \\ \mathbf{y}(0,x) = \mathbf{y}_0(x) & \text{on } \Omega. \end{array}\right\} \tag{3}$$

We find a necessary condition on the pressure for the approximate controllability of the problem (see Theorem 2 and Remark 2). Finally, in Section 3, we study the approximate controllability of the Burgers problem under suitable constraints on the desired state $y_d(x)$. We recall some previous results by El Badia and Ain Seba (1992) and Fursikov and Imanuvilov (1993b) on the exact controllability for suitable desired states y_d. Using the last of those references we give a L^p-approximate controllability for a larger class of desired states. We conjecture that some sharper results can be found following the ideas of Díaz (1994b). We start here such a long programme by proving the L^p-approximate controllability for the general quasilinear problem

$$\left.\begin{array}{ll} y_t - \Delta y + \operatorname{div}\mathbf{B}(y) = u\chi_\omega & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x) & \text{on } \Omega, \end{array}\right\} \tag{4}$$

where Ω denotes an open bounded regular set of $\mathbb{R}^N$ and ω is an open subset of Ω. Here $\mathbf{B} \in \mathcal{C}(\mathbb{R}:\mathbb{R}^N)$ is assumed to be differentiable at some $s_0 \in \mathbb{R}$ and sublinear at the infinity, *i.e.* there exists $M > 0$ such that

$$|\mathbf{B}(s)| \leq c_1 + c_2|s| \qquad \text{for any } s\in\mathbb{R},\ |s| > M. \tag{5}$$

The remaining parts of the programme introduced in Díaz (1994b) will be developped for the Burgers problem elsewhere.

2 OBSTRUCTION FOR THE BURGERS EQUATION

The main goal of this section is to show that the approximate controllability in $\mathcal{X} = L^p(0,L)$, $1 \leq p \leq \infty$, or $\mathcal{X} = \mathcal{C}([0,L])$ can not holds for the Burgers problem (1). To do that we shall prove the existence of an *universal obstruction function* which will provide an explicit bound for any element of the attainable set $R_T := \{y(T,\cdot;u) : u \in \mathcal{U},\ y \text{ solution of } (1)\}$, where $\mathcal{U}$ denotes the space of controls.

We start by recalling that the existence and uniqueness of solutions of problem (1) can be obtained in different ways. So, for instance, the results of Alt and Luckhaus (1983) can be applied showing the existence and uniqueness of a weak solution $y \in \mathcal{C}([0,T] : L^2(0,L))\cap L^\infty((0,T)\times(0,L))\cap L^p(0,T: W^{1,p}(0,L))$ for any $p\in[1,\infty)$ with $y_t \in L^{p'}(0,T: W^{-1,p'}(0,L))$ assumed $u \in \mathcal{U} := W^{1,1}(0,T)$. In fact, it is not difficult to show that this solution is much more regular.

The existence of the mentioned *universal obstruction function* is a consequence of the presence of the superlinear term $yy_x = (y^2)_x/2$ at the equation. Such a function will be built as solution of the following problem

$$\left.\begin{array}{ll} Y_t + YY_x - \nu Y_{xx} = 0 & \text{in } (0,T)\times(0,L) \\ Y(t,0) = 0,\quad Y(t,L) = +\infty, & t\in(0,T) \\ Y(0,x) = y_0(x), & x\in(0,L). \end{array}\right\} \tag{6}$$

THEOREM 1
(i) *Problem* (6) *has a minimal solution* $\underline{Y} \in \mathcal{C}([0,T]:L^2(0,L-\varepsilon))\cap L^p(0,T: W^{1,p}(0,L-\varepsilon))$, *for any* $\varepsilon \in (0,L)$ *and for any* $1 \leq p < \infty$. *Moreover, there exists some positive constants* C_1, C_2 *only dependent on* L,ν *and* $\|y_0\|_\infty$ *such that*

$$\underline{Y}(t,x) \leq C_1\left(\frac{1}{L-x} - \frac{1}{L}\right) + C_2. \tag{7}$$

(ii) *Let* $y(t,\cdot;u)$ *be the solution of the associated problem* (1). *Then*

$$y(t,x;u) \leq \underline{Y}(t,x) \quad \textit{for any } t\in[0,T] \textit{ and a.e. } x\in(0,L). \tag{8}$$

In particular the approximate controllability property can not be satisfied in the set of states $\mathcal{X} = L^q(0,L)$, $1 \leq q \leq \infty$.

PROOF. Given $n \in \mathbb{N}$ we consider the truncated problem

$$\left.\begin{array}{ll} Y_t + YY_x - \nu Y_{xx} = 0 & \text{in } (0,T)\times(0,L) \\ Y(t,0) = 0,\quad Y(t,L) = n, & t\in(0,T) \\ Y(0,x) = y_0^n(x) := \min\{y_0(x),n\}, & x\in(0,L). \end{array}\right\} \tag{9}$$

The existence and uniqueness of the solution Y_n of (9) holds as mentioned before. Furthermore, the comparison principle is satisfied (see *e.g.* Alt and Luckhaus (1983)) and so we have

$$Y_1 \leq Y_2 \leq \cdots \leq Y_n \leq Y_{n+1} \leq \cdots \quad \text{on } (0,T)\times(0,L).$$

In order to construct a global barrier function we define

$$W(x) := K_1(L-x)^{-\alpha} + K_2, \quad x\in(0,L) \tag{10}$$

where $K_1, \alpha > 0$ and $K_2 \geq 0$ will be choosen later. Then

$$WW_x - \nu W_{xx} = \alpha K_1^2 (L-x)^{-2\alpha-1} + K_1 K_2 \alpha (L-x)^{-\alpha-1} - \nu\alpha(\alpha+1) K_1 (L-x)^{-\alpha-2}.$$

So, making $\alpha = 1$ we find that

$$WW_x - \nu W_{xx} = (-2\nu + K_1) K_1 (L-x)^{-3} + K_1 K_2 (L-x)^{-2} := f(x). \tag{11}$$

Then, for any $n \in \mathbb{N}$, we have

$$\left.\begin{array}{ll} Y_{n,t} + Y_n Y_{n,x} - \nu Y_{n,xx} = 0 \leq WW_x - \nu W_{xx} & \text{in } (0,T)\times(0,L) \\ Y_n(t,0) = 0 \leq W(0), \quad \limsup_{x\uparrow L}(Y_n(t,x) - W(x)) < 0, & t \in (0,T) \\ Y_n(0,x) \leq \|y_0\|_\infty \leq W(x), & x \in (0,L) \end{array}\right\}$$

assuming that

$$K_1 \geq 2\nu, \quad K_2 \geq 0 \qquad \text{and} \qquad \|y\|_\infty \leq K_1 L^{-1} + K_2. \tag{12}$$

In particular, if

$$K_1 = 2\nu \qquad \text{and} \qquad K_2 \geq \left[\|y_0\|_\infty - 2\nu L^{-1}\right]_+ \tag{13}$$

then (12) holds, $f \in L^\infty(0, L-\varepsilon)$, for any $\varepsilon \in (0,L)$, and from the comparison principle we deduce that $Y_n(t,x) \leq W(x)$ for any $n \in \mathbb{N}$, $t \in [0,T]$ and *a.e.* $x \in (0,L)$. By the Beppo-Levi Theorem we have that $Y_n(t,\cdot) \to \underline{Y}(t,\cdot)$ in $L^p(0, L-\varepsilon)$, for any $\varepsilon \in (0,L)$ and any $1 \leq p \leq \infty$. Using straightforward arguments it is easy to see that $\underline{Y}$ is the minimal solution of problem (6) and that (7) holds. Part (ii) comes from the fact that if u is a bounded function then choosing $n_0 \in \mathbb{N}$ such that $n_0 \geq \max\{\|y_0\|_\infty, \|u\|_\infty\}$, we have $y(t,x;u) \leq Y_n(t,x)$ for any $n \geq n_0$, $t \in [0,T]$ and *a.e.* $x \in (0,L)$. Finally, if the desired state $y_d(x)$ is such that $y_d(x) > W(x)$ on an open subset of $(0,L)$ then condition (3) fails, for any $u \in \mathcal{U}$, for any $\varepsilon > 0$. ■

REMARK 1

1. The fact that (1) is not approximately controllable can also be proved by using an *universal integral estimate* as obtained in Fursikov and Imanuvilov (1993) by multiplying the equation by $(b-x)^n y_+^3(t,x)$ with $b \in (0,L)$ and $n > 5$, integrating by parts and applying Hölder and Young inequalities. Such a method was already introduced by A. Bamberger for the study of superlinear semilinear parabolic equations (see Henry (1978)). A more sophisticated energy method can also be applied to higher order superlinear parabolic equations: see Bernis, Díaz and Ramos (1995).
2. The obstruction phenomenon also holds for other nonlinear parabolic controls problems with superlinear terms in the equations such as $y_t - \Delta y + \lambda |y|^{p-1} y = 0$ ($p > 1$: superlinear semilinear equation), $y_t - \Delta |y|^{m-1} y = 0$ ($m > 1$: the porous media equation) and $y_t - \mathrm{div}\,(|\nabla y|^{p-2}\nabla y) = 0$ ($p > 2$: Non-Newtonian flows). See Díaz (1991a).
3. The study of boundary value problems blowing-up on a part of the boundary has been largely considered by many authors: Bieberbach, Rademacher, Keller, P.L.Lions and Lasry, *etc.* (see references in Bandle and Marcus (1990), G.Díaz and Letelier (1993) and Bandle, Díaz and Díaz (1994)).
4. We point out that the conclusion of Theorem 1 holds if u is not bounded (truncate u and pass to the limit). It also holds for solutions of the non-homogeneous equation

$$y_t + yy_x - \nu y_{xx} = f(t,x) \tag{14}$$

and the rest of conditions as indicated in (1), assuming that $f \in L^1((0,T)\times(0,L-\varepsilon))$ for any $\varepsilon \in (0,L)$ and

$$f(t,x) \leq M_1 (L-x)^{-3} + M_2 (L-x)^{-2} \qquad \text{for any } (t,x) \in (0,T)\times(0,L) \tag{15}$$

for some nonnegative constants M_1 and M_2.
5. Similar results can be obtained for other control problems associated to the Burgers equation. So, for the case in which the control acts on the left boundary, *i.e.* $y(t,0) = u(t)$, $y(t,L) = 0$, $t \in (0,T)$ and the rest of conditions of (1) an inequality (analogous to (8)) holds

$$\frac{c_1}{x} - c_2 \leq y(t,x;u) \quad \text{for any } t \in (0,T),\ a.e.\ x \in (0,L) \tag{16}$$

for any bounded function u and for some constants c_1 and c_2. Analogously, consider the controllability from the interior problem

$$\left.\begin{array}{ll} y_t + yy_x - \nu y_{xx} = u\chi_\omega & \text{in } (0,T)\times(0,L) \\ y(t,0) = y(t,L) = 0, & t \in (0,T) \\ y(0,x) = y_0(x), & x \in (0,L), \end{array}\right\} \tag{17}$$

where $\omega = (a,b)$ is an open subinterval of $(0,L)$ and $u \in L^2(\omega)$. An easy application of the arguments of Theorem 1 shows that

$$y(t,x;u) \leq c_1\left(\frac{1}{a-x} - \frac{1}{b}\right) + c_2 \qquad \text{for any } t \in [0,T] \text{ and } a.e.\ x \in (0,a) \tag{18}$$

for any $u \in L^2(0,T)\times\omega)$ and for some constants c_1 and c_2. □

We shall end this section by applying Theorem 1 to the study of the approximate controllability of the Navier-Stokes system. Such a question was already raised by Lions (1990) and still remains an open question. Our contribution will be limited to give a necessary condition on the pressure assuming that the approximate controllability holds. For the sake of simplicity in the exposition we shall merely deal with the case of a planar flow occupying a rectangle $\Omega = (0,L_1)\times(0,L_2)$ (see problem (3) in the Introduction). Many other domains and other control problems can be considered analogously.

THEOREM 2

Let $\mathbf{y}_0 \in L^\infty(\Omega)^2$. *Consider* $\varepsilon > 0$ *and* $\mathbf{y}_d = (y_{d,1}, y_{d,2}) \in L^2(\Omega)^2$ *such that there exists a positively measured subset* P *of* Ω *and a constant* $C \geq [\|y_{0,1}\|_\infty - \nu L_1]_+$ *such that*

$$\|[y_{1,d}(x) - \nu(L-x)^{-1} - C]_+\|_{L^2(P)} > \varepsilon. \tag{19}$$

Let $\mathbf{u} = (u_1,u_2)$ *be any control such that* $\|\mathbf{y}(T,\cdot;\mathbf{u}) - \mathbf{y}_d\|_{L^2(\Omega)^2} \leq \varepsilon$. *Then necesarily*

$$-\frac{\partial p}{\partial x_1}(t,x_1,x_2;\mathbf{u}) > 2\nu C(L_1 - x_1)^{-2} \tag{20}$$

on some positively measured set $M \subset (0,T)\times\Omega$.

The proof will use some comparison results between the solution of a Burgers problem (1) and the first component of the solution of the Navier-Stokes system (4). A first result in this direction is the following

LEMMA 1

Let $z \in C^{1,2}_{t,x}((0,T)\times(0,L_1))\cap C([0,T]\times[0,L_1))$ *and assume that* $y=(y_1,y_2)$ *is a classical solution of* (3). *Let* $z \in C([0,T]\times(0,L_1))$ *be such that*

$$\left.\begin{array}{ll} z_t+zz_{x_1}-\nu z_{x_1x_1}=f(t,x_1) & \text{in } (0,T)\times(0,L_1) \\ z(t,0)=0, \quad \liminf_{x\uparrow L_1} z(t,x)>\|u_1(t,\cdot)\|_{L^\infty(\Gamma)}, & t\in(0,T), \\ z(0,x_1)\geq \|y_{0,1}\|_{L^\infty(\Omega)}, & x_1\in(0,L_1) \end{array}\right\} \tag{21}$$

where $f \in C((0,T)\times(0,L_1))$ *verifies*

$$f(t,x_1)\geq -\frac{\partial p}{\partial x_1}(t,x_1,x_2;u) \qquad \text{for } (t,x_1,x_2)\in(0,T)\times\Omega \tag{22}$$

(*the control* $u=(u_1,u_2)$ *in* (3) *is here assumed to be known*). *Finally, assume that*

$$z_{x_1}(t,x_1)>0 \qquad \text{for any } (t,x_1)\in(0,T)\times(0,L_1). \tag{23}$$

Then

$$z(t,x_1)\geq y_1(t,x_1,x_2;u) \qquad \text{for any } t\in[0,T] \text{ and } (x_1,x_2)\in\Omega. \tag{24}$$

PROOF. From (21) and (23) we can assume, without loss of generality, that there exists $t_0\in(0,T]$ and $(x_{0,1},x_{0,2})\in\Omega$ such that

$$\sup_{[0,T]\times\overline{\Omega}}(y_1(t,x_1,x_2)-z(t,x_1)) = y_1(t_0,x_{1,0},x_{2,0})-z(t,x_{1,0})>0$$

(otherwise (24) holds). Define $w(t,x_1,x_2):=y_1(t,x_1,x_2)-z(t,x_1)$. Then

$$\frac{\partial y_1}{\partial x_1}(t_0,x_{0,1},x_{0,2})=\frac{\partial z}{\partial x_1}(t_0,x_{0,1})>0 \quad \text{and} \quad \frac{\partial y_1}{\partial x_2}(t_0,x_{0,1},x_{0,2})=0,$$
$$\Delta w(t_0,x_{0,1},x_{0,2})\leq 0 \qquad (\text{ i.e. } \Delta y_1(t_0,x_{0,1},x_{0,2})\leq z_{x_1x_1}(t_0,x_{0,1}))$$

and

$$w_t(t_0,x_{1,0},x_{2,0})\geq 0.$$

On the other hand

$$w_t=-y_1y_{1,x_1}-y_2y_{1,x_2}+\nu\Delta y_1-\nu z_{x_1x_1}-p_{x_1}-f+zz_{x_1}.$$

Thus

$$w_t(t_0,x_{0,1},x_{0,2})\leq -w(t_0,x_{0,1},x_{0,2})z_{x_1}(t_0,x_{0,1})<0$$

which is a contradiction. ∎

A more general result, obtained by replacing (23) by a nonnegativeness condition, and valid for bounded weak solutions of the Navier-Stokes problem, can be proved by introducing an artificial constant in the Burgers equation.

LEMMA 2

Let $z\in C([0,T]: L^2(0,L_1))\cap L^2(0,T:H^1(0,L_1))\cap W^{1,1}(0,T:L^1(0,L_1))$ *be a nonnegative function satisfying*

$$\left.\begin{array}{ll} z_t+2zz_{x_1}-\nu z_{x_1x_1}=f(t,x_1) & \text{in } (0,T)\times(0,L_1) \\ z(t,0)=0, \quad z(t,L_1)\geq\|u_1(t,\cdot)\|_{L^\infty(\Gamma)}, & t\in(0,T) \\ z(0,x_1)\geq\|y_{0,1}\|_{L^\infty(\Omega)}, & x_1\in(0,L_1). \end{array}\right\} \tag{25}$$

Let $u\in\mathcal{U}\subset L^\infty((0,T)\times\Gamma)^2$ *be given such that* (3) *has a unique bounded weak solution* $y(t,\cdot;u)=(y_1(t,\cdot;u),y_2(t,\cdot;u))$ *of pressure* $p=p(t,\cdot;u)$. *Assume* $f\in L^1((0,T)\times(0,L_1))$ *satisfying* (22). *Then*

$$z(t,x_1)\geq y_1(t,x_1,x_2:u) \quad \text{for any } t\in[0,T] \text{ and a.e. } (x_1,x_2)\in\Omega.$$

PROOF. We start by noting that from the incompressibility condition $\operatorname{div} y=0$ we deduce that

$$(y\cdot\nabla)y_1=y_1y_{1,x_1}+y_2y_{1,x_2}=\left(\frac{1}{2}y_1^2\right)_{x_1}+(y_1y_2)_{x_2}-y_1y_{2,x_2}$$
$$=\left(\frac{1}{2}y_1^2\right)_{x_1}+(y_1y_2)_{x_2}+y_1y_{1,x_1}=\operatorname{div}(y_1^2,y_1y_2).$$

Then

$$\begin{array}{ll} z_t+\operatorname{div}(z_1^2,y_2z)-\nu\Delta z\geq y_{1,t}+\operatorname{div}(y_1^2,y_2y_1)-\nu\Delta y_1 & \text{in } (0,T)\times\Omega, \\ z\geq y_1 & \text{on } (0,T)\times\partial\Omega \\ z(0,\cdot)\geq y_1(0,\cdot) & \text{on } \Omega. \end{array}$$

Finally the comparison theorem of Díaz and de Thelin (1994) (see Theorem 3) can be applied since the function $K:\ (0,T)\times\Omega\times\mathbb{R}\to\mathbb{R}^2$ defined by $K(t,x_1,x_2,r)=(r^2,y_2(t,x_1,x_2)r)$ generates a locally Lipschitz functional on $L^1(\Omega)$ for any fixed $t\in[0,T]$. ∎

PROOF of THEOREM 2. Assume, by contrary, that $-p_{x_1}(t,x_1,x_2:u)\leq 2\nu(L_1-x)^{-2}$ *a.e.* on $(0,T)\times\Omega$. Taking $z(t,x_1)=\nu(L_1-x)^{-1}+C$ and arguing as in Theorem 1 it is easy to see that Lemma 2 can be applied on $(0,L_1-\delta)$ for any $\delta>0$ small enough. In consequence we have $y_1(t,x_1,x_2;u)\leq z(t,x_1)$ for any $t\in[0,T]$ and *a.e.* $(x_1,x_2)\in\Omega$. Therefore

$$\|y(T,\cdot;u)-y_d\|_{L^2(\Omega)^2}\geq\|y_1(T,\cdot;u)-y_{1,d}\|_{L^2(\Omega)}$$
$$\geq\|[y_1(T,\cdot;u)-y_{1,d}]_+\|_{L^2(P)}+\|[y_{1,d}-y_1(T,\cdot;u)]_+\|_{L^2(P)}$$
$$\geq\left\|\left[y_{1,d}-\nu(L-x)^{-1}-C\right]_+\right\|_{L^2(P)}>\varepsilon$$

which is a contradiction. ∎

REMARK 2

Roughly speaking Theorem 2 says that if $y_{d,1}(x_1,x_2)$ is big enough near the boundary $x_1=0$ then necessarily $p_{x_1}(t,x_1,x_2)$ must be "very negative" on some part of $(0,T)\times\Omega$. Notice that although we do not know if this is our case, there are many explicit solutions of the Navier-Stokes system on special domains having $p_{x_i}=0$ for some i and that this would contradict the necessary condition for the approximate controllability. □

3 SOME CONTROLLABILITY RESULTS: THE BURGERS EQUATION AND A RELATED QUASILINEAR EQUATION

Although the obstruction phenomenon shows that the approximate controllability can not hold, some partial results in this direction can be obtained. So, El Badia and Ain Seba (1992) use some variants of the Hopf-Cole transformation Φ to prove the exact controllability of problems (1) and (17) when the desired state y_d belongs to the transformed set $\Phi(E)$ where E is the set of attainable states associated to the linear heat equation.

A different approach was followed by Fursikov and Imanuvilov (1993b). They prove several results on the attainable set associated to the Burgers equation. Their main result, stated for the formulation given in (1) says the following: Let $y_0 \in \mathrm{H}^1(0,L)$ arbitrary and let $\hat{y}(t,x)$ be any solution of the Burgers equation satisfying $\hat{y}(t,0)=0$ for $t\in(0,T)$ and $\hat{y}(0,x)=y_0(x)$ on $(0,L)$ but without any prescription at $x=L$. Define $y_d(x):=\hat{y}(T,x)$. Then there exists $\delta_0>0$ such that for any $\delta\in(0,\delta_0)$ there exists $y_0^*\in\mathrm{H}^1(0,L)$ with $\|y_0-y_0^*\|_{\mathrm{H}^1}\leq\delta$ and a control $u\in\mathcal{C}(0,T)$ such that if $y^*(t,x)$ is the solution of (1) associated to the data y_0^* and u then $y^*(T,x)=\hat{y}(T,x)$ on $(0,L)$.

With the help of the above result we can prove a L^p-approximate controllability criterium for a larger class of desired states.

PROPOSITION 1

Let $y_0\in\mathrm{L}^\infty(0,L)$ and let $y_d(\cdot)=\tilde{y}(T,\cdot)$ where $\tilde{y}\in\mathcal{C}([0,T]:L^\infty(0,L))$ is any solution of the Burgers equation satisfying $\tilde{y}(t,0)=0$ for $t\in(0,T)$ and $\tilde{y}(0,x)=y_0(x)$ on $(0,L)$. Then, for any $\varepsilon>0$ there exists a control $u\in\mathcal{C}(0,T)$ such that for any p, $1\leq p\leq\infty$, we have

$$\|y(T,\cdot;u)-y_d\|_{\mathrm{L}^p(0,L)}\leq\varepsilon$$

PROOF. By regularizing y_0 we can find $\bar{y}_0\in\mathrm{H}^1(0,L)$ such that $\|y_0-\bar{y}_0\|_\infty\leq\varepsilon/2$. On the other hand, by applying the T-accretiveness of the operator $-y_{xx}+yy_x$ on $\mathrm{L}^\infty(0,L)$ (*i.e.* by a generalization of the classical maximum principle, see, *e.g.* Benilan (1981)) we know that if $y_i(t,x)$, $i=1,2$, are solutions of problem (1) associated to the initial data $y_{0,i}(x)$ ($y_{0,i}\in\mathrm{L}^\infty(0,L)$) and to the same boundary datum $u(t)$ then we have

$$\|y_1(t,\cdot)-y_2(t,\cdot)\|_\infty\leq\|y_{0,1}(\cdot)-y_{0,2}(\cdot)\|_\infty \tag{26}$$

for any $t\in[0,T]$. Let now $\bar{y}(t,x)$ be the solution of the Burgers equation for initial datum $\bar{y}_0$ and boundary conditions $\bar{y}(t,0)=0$, $\bar{y}(t,L)=\tilde{y}(t,L)$ for $t\in(0,T)$. Thus, by virtue of (26) we have $\|\bar{y}(t,\cdot)-\tilde{y}(t,\cdot)\|_\infty\leq\varepsilon/2$ for any $t\in[0,T]$. Now, by the mentioned result by Fursikov and Imanuvilov (1993b) we deduce the existence of a control function $u\in\mathcal{C}(0,T)$ and $y_0^*\in\mathrm{H}^1(0,L)$ with $\|\bar{y}_0-y_0^*\|_{\mathrm{H}^1}\leq\min(\delta_0/2,\varepsilon/2)$ such that if $y^*(t,x)$ is the solution of (1) associated to u and y_0^* we have $y^*(T,x)=\bar{y}(T,x)$ on $(0,L)$. Finally, if $y(t,x;u)$ is the solution of (1) associated to $u(t)$ and $y_0(x)$, by virtue of (26) we have

$$\|y(T,\cdot)-y_d(\cdot)\|_\infty\leq\|y(T,\cdot)-\bar{y}(T,\cdot)\|_\infty+\|\bar{y}(T,\cdot)-y^*(T,\cdot)\|_\infty$$
$$+\|y^*(T,\cdot)-\tilde{y}(T,\cdot)\|_\infty\leq\varepsilon/2+\min(\delta_0/2,\varepsilon/2)\leq\varepsilon.$$ ∎

Notice that if we define the class of functions

$$\mathcal{Y}:=\{w\in\mathcal{C}((0,L)):\ w(x)\leq\underline{Y}(T,x)\quad\text{for any } x\in(0,L)\}$$

where $\underline{Y}$ is the universal obstruction function given in Theorem 1 then, arguing as in the proof of Theorem 1, it is clear that the desired state y_d considered in Proposition 1 satisfies that $y_d\in\mathcal{Y}$. Notice also that part (ii) of Theorem 1 proves that if, for instance, we know that $y_d\in\mathcal{C}(0,L)$ satisfies the $\mathcal{C}^0$-approximate controllability property then, necessarily, $y_d\in\mathcal{Y}$. In fact, we conjecture that this necessary condition is also sufficient. Such type of results was obtained in the case of superlinear semilinear problems in Díaz (1994b). The proof there was established in two different steps: a) truncation of the nonlinear term and application of approximate controllability results for *sublinear* semilinear equations, and b) obtention of a priori estimates on the control (associated to the truncated problem) independent of the truncation value $n\in\mathbb{N}$.

A first difficulty to apply such a programme for the case of the Burgers equation is that, as far as we know, there is not any controllability result for the associated sublinear case available in the literature. Due to that, we shall start here the mentioned programme by considering the question of the L^2-approximate controllability for a general class of quasilinear parabolic problems including the case mentioned above. For the sake of simplicity in the exposition we replace the boundary controllability considered in (1) for an internal controllability formulation. More precisely, let Ω be an open bounded regular set of $\mathbb{R}^N$ and ω be an open subset of Ω. Given $y_0\in\mathrm{L}^2(\Omega)$ we consider the control problem (4) mentioned in the Introduction.

The existence of a solution $y\in\mathcal{C}([0,T]:\ \mathrm{L}^2(\Omega))$ (when the control $u\in\mathrm{L}^2((0,T)\times\omega)$ is given) can be obtained by different methods (see *e.g.* Alt and Luckhaus (1983)). The uniqueness of solutions is a more delicate question. It was established under the additional assumption

$$\mathrm{B}\in\mathcal{C}^{0,\alpha}(\mathbb{R}:\mathbb{R}^N),\ \text{(the space of Hölder continuous functions), and } \alpha\geq\frac{1}{2}$$

by several authors: Alt and Luckhaus, Artola, Chipot and Rodrigues, Gagneux and Guerfi, Díaz and de Thelin (see references in Díaz and de Telin (1994)). More recently, the uniqueness of the solution has been established for merely continuous functions by Gagneux and Madaune-Tort (1994) using previous ideas introduced by Carrillo (1986) for the elliptic case. Concerning the approximate controllability we have

THEOREM 3

Under condition (5) *on* B *problem* (4) *is approximately controllable in* $\mathrm{L}^2(\Omega)$.

As usual, we shall prove Theorem 3 through a fixed point argument applied to an operator associated to a linearised problem. So we consider, previously, the following problem

$$\left.\begin{array}{ll} y_t-\mathcal{L}(t)y=u\chi_\omega & \text{in } (0,T)\times\Omega \\ y=0 & \text{on } (0,T)\times\partial\Omega \\ y(0,x)=y_0 & \text{on } \Omega \end{array}\right\} \tag{27}$$

where

$$\mathcal{L}(t)y=\Delta y-\operatorname{div}\,(\mathrm{b}(t,\cdot)y) \tag{28}$$

assuming (for simplicity) $\mathrm{b}\in\mathrm{L}^\infty((0,T)\times\Omega)$. The existence and uniqueness of the solutions $y\in\mathcal{C}([0,T]:\ \mathrm{L}^2(\Omega))\cap\mathrm{L}^2(0,T:\mathrm{H}_0^1(\Omega))$, for a given data $y_0\in\mathrm{L}^2(\Omega)$ and $u\in\mathrm{L}^2((0,T)\times\omega)$

can be obtained, for instance, by applying the theory of abstract operators $\mathcal{A}(t)$ such that $\mathcal{A}(t)+\lambda I$ are maximal monotone operators on $L^2(\Omega)$ for some $\lambda>0$ and *a.e.* $t\in(0,T)$ (see *e.g.* Brezis (1973)). We point out that, in general, λ cannot be taken as $\lambda=0$ if $b\neq 0$ (see Lemma 8.4. of Gilbarg and Trudinger (1977)).

PROPOSITION 2

When u spans $L^2((0,T)\times\omega)$, $y(T,\cdot;u)$ *spans an affine subspace which is dense in* $L^2(\Omega)$.

PROOF. We follows the arguments of Lions (1968). Without loss of generality we can assume that $y_0\equiv 0$. Let $\varphi_0\in L^2(\Omega)$ such that

$$\int_\Omega y(T,x;u)\varphi_0(x)dx=0 \qquad \text{for any } u\in L^2((0,T)\times\omega). \tag{29}$$

Let $\mathcal{L}^*$ be the *formal adjoint* operator of $\mathcal{L}$, *i.e.* $\mathcal{L}^*(t)\varphi=\Delta\varphi+b(t,\cdot)\cdot\nabla\varphi$ (see Gilbarg and Trudinger (1977) p. 172). Define φ as the solution to the backwards problem

$$\left.\begin{array}{ll} -\varphi_t-\mathcal{L}^*(t)\varphi=0 & \text{in } (0,T)\times\Omega \\ \varphi=0 & \text{on } (0,T)\times\partial\Omega \\ \varphi(T,x)=\varphi_0(x) & \text{on } \Omega. \end{array}\right\} \tag{30}$$

Applying Green's formula we get

$$\int_\Omega y(T,x;u)\varphi_0(x)dx=\int_0^T\int_\omega \varphi(t,x)u(t,x)dxdt. \tag{31}$$

Therefore (29) implies that $\varphi=0$ in $(0,T)\times\omega$. Finally, applying the unique continuation property (see Corollary 1.2 of Saut and Scheurer (1987), valid for non necessarily selfadjoint operators) we conclude that $\varphi\equiv 0$ in $(0,T)\times\Omega$ and the conclusion holds by the Hahn-Banach theorem. ∎

In order to consider the nonlinear problem (4) we shall need more information on the application $y_d\mapsto u$ (for a fixed $\varepsilon>0$) found in Proposition 2. In fact this is a multivalued map since it is easy to see that there are infinitely many $u\in L^2((0,T)\times\Omega)$ satisfying

$$\|y(T,\cdot;u)-y_d\|_{L^2(\Omega)}\leq\varepsilon, \tag{32}$$

where $y(t,\cdot;u)$ denotes the solution of (27). We shall follow now some direct methods introduced in Lions (1992a), (1992b) and later generalized and improved in Fabré, Puel and Zuazua (1992a), (1992b) leading to the existence of "quasi bang-bang controls". The results of this last reference can be modified easily to our context although in the mentioned work it is always assumed that $\mathcal{L}^*=\mathcal{L}=\Delta$. Given $\varphi_0\in L^2(\Omega)$ we define the functional

$$\mathcal{J}(\varphi_0)=\frac{1}{2}\left(\int_0^T\int_\omega|\varphi(t,x)|dxdt\right)^2+\varepsilon\|\varphi_0\|_{L^2(\Omega)}-\int_\Omega y_d(x)\varphi_0(x)dx.$$

Following Fabré, Puel and Zuazua (1992b) this functional is continuous from $L^2(\Omega)$ into $\mathbb{R}$, strictly convex and satisfies the coerciveness condition

$$\liminf_{\|\varphi_0\|\to\infty}\frac{\mathcal{J}(\varphi_0)}{\|\varphi_0\|_2}\geq\varepsilon$$

(this last condition is proved from the unique continuation property obtained in Saut and Scheurer (1987)). Therefore there is a unique $\psi_0\in L^2(\Omega)$ such that

$$\mathcal{J}(\psi_0)=\min_{\varphi_0\in L^2(\Omega)}\mathcal{J}(\varphi^0). \tag{33}$$

It is easy to see that, in fact, $\psi_0=0$ *iff* $\|y_d\|_2\leq\varepsilon$. By studying the associated Euler-Lagrange (multivalued) equation it is proved the following result.

PROPOSITION 3

Let ψ_0 *be the solution of* (33) *and let* $\psi(t,x)$ *be the solution of* (30) *associated to* $\varphi_0=\psi_0$. *Then there exists* $v\in\operatorname{sign}(\psi)\chi_\omega$ *such that the control* $u:=(\|\psi\|_{L^1((0,T)\times\omega)})v$ *leads to a solution* $y(t,\cdot;u)$ *of* (27) *such that* $\|y(T,\cdot;u)-y_d\|\leq\varepsilon$.

REMARK 3

The operator $\mathcal{L}$ defined in (28) can be taken, more generally, of the form

$$\mathcal{L}(t)y=\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial y}{\partial x_j}\right)+\sum_{i=1}^{N}\frac{\partial}{\partial x_i}(b_i(t,x)y) \tag{34}$$

with $a_{ij}\in\mathcal{C}^1((0,T)\times\Omega)$ satisfying

$$\sum_{i,j=1}^{N}a_{ij}(t,x)\xi_i\xi_j\geq\alpha(x,t)|\xi|^2,\qquad\forall(t,x)\in(0,T)\times\Omega,\ \forall\xi\in\mathbb{R}^N$$

for some $a(x,t)>0$ and $b=(b_1,\ldots,b_N)\in L^\infty((0,T)\times\Omega)$. We point out that in Glowinski and Lions (1994) the nondivergential form operator

$$\mathcal{L}y=\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}\right)+V_0\cdot\nabla y \tag{35}$$

was considered under the assumption $V_0\in L^\infty(\Omega)^N$ and $\operatorname{div}V_0=0$ on Ω. Notice that under this assumption the operator can be written as in (34) (use that $\operatorname{div}(V_0y)=\operatorname{div}V_0y+V_0\cdot\nabla y$). Finally, we point out that if b is a $W^{1,\infty}(\Omega)^N$ function, independent on t, then the result of Lin (1991) allows to know that the set $\{(t,x):\ \psi(t,x)=0,\ \psi \text{ solution of } (30)\}$ has zero Lebesgue measure and so the controls u, in Proposition 2, are of type bang-bang. □

PROOF of THEOREM 3. Let us assume that B is differentiable at $s_0=0$. The case $s_0\neq 0$ can be easily treated by an homogeneization argument. Define the function $g:\mathbb{R}\to\mathbb{R}^N$ by

$$g(s)=\frac{B(s)-B(0)}{s}\qquad\text{if } s\neq 0 \text{ and } g(0)=B'(0).$$

From the regularity of B and (5) it is clear that $g\in\mathcal{C}(\mathbb{R}:\mathbb{R}^N)\cap L^\infty(\mathbb{R}:\mathbb{R}^N)$. Given $z\in L^2((0,T)\times\Omega)$, we define $b=-g(z)\in L^\infty((0,T)\times\Omega)$ and so, by Proposition 2, the associated linear problem (27) is approximately controllable. More precisely, given $y_d\in L^2(\Omega)$ and $\varepsilon>0$ let $u(z)$ be the control and $y^*(t,\cdot;z)$ the solution of (27) mentioned in Proposition 3. Define now the nonlinear mapping $\Lambda:\ L^2(\Omega)\to\mathcal{P}(L^2(\Omega))$ by $\Lambda(z)=$

$\{y^*(t,\cdot;u(z))$ for some $u(z) \in \|\psi\|_1 \mathrm{sign}(\psi)\chi_\omega\}$. Notice that Λ can be multivalued since the uniqueness of $u(z)$ is not assured. It is easy to see that any fixed point of Λ allow us to obtain the conclusion of the theorem. In order to apply the Kakutani fixed point theorem (in the weak form given in Aubin (1984)) we need to check the following conditions: (i) $\forall z \in L^2((0,T)\times\Omega)$, the set $\Lambda(z)$ is non-empty, convex and compact in $L^2((0,T)\times\Omega)$, (ii) $\Lambda(z)$ is upper hemi-continuous. The proof of both properties can be obtained by adapting the arguments of Fabré, Puel and Zuazua (1992b). The convexity of $\Lambda(z)$ is a consequence of the linearity of (27) and the convexity of the set $\{v \in L^2((0,T)\times\Omega) : v \in \mathrm{sign}(\psi)\chi_\omega\}$. As $g(z)$ is uniformly bounded in $L^\infty((0,T)\times\Omega)$ we can prove the existence of a compact subset $\mathcal{K} \subset L^2((0,T)\times\Omega)$ such that $\Lambda(z) \subset \mathcal{K}$ for any $z \in L^2((0,T)\times\Omega)$. To prove this we first notice that $\{y^*(\cdot,\cdot;z)\}$ and $\{y_t^*(\cdot,\cdot;z)\}$ remain uniformly bounded in $L^2(0,T:H_0^1(\Omega))$ and $L^2(0,T:H^{-1}(\Omega))$ respectively when z runs $L^2((0,T)\times\Omega)$ (this can be obtained multiplying by $y^*(z)$, integrating by parts and applying the coerciveness of $\mathcal{L}(t)$). Then, by well-known results (see *e.g.* Corollary 4 of Simon (1987)), we conclude that $\{y^*(\cdot,\cdot;z)\}$ is relatively compact in $L^2((0,T)\times\Omega)$. Choosing as $\mathcal{K}$ the closure of this set in $L^2((0,T)\times\Omega)$ the proof of property (i) is reduced to show that $\Lambda(z)$ is a closed set. This is proved without any difficulty using that the multivalued (maximal monotone) graph $\mathrm{sign}(\cdot)$ is strongly-weakly closed, the coerciveness of $\mathcal{L}(t)$ and the compactness of the Green operator associated to (27). The proof of (ii) follows as in Fabré, Puel and Zuazua (1992b) once that we already know the compactness of the set $\mathcal{K}$ and the continuity of function g. ■

REMARK 4

Theorem 3 can be improved in several directions. First of all, following closely the arguments of Fabré, Puel and Zuazua (1992b), the approximate controllability property can be also obtained on the spaces $L^p(\Omega)$, $1 \le p < \infty$, and $\mathcal{C}^0(\overline{\Omega})$. On the other hand, the approximate controllability can be obtained for a more general class of nonlinear equations of the type

$$y_t - \Delta y + \mathrm{div}\ (\mathrm{B}(y)) + f(y) = u\chi_\omega$$

assuming B as before and f be a continuous real function, differentiable at some $s_1 \in \mathbb{R}$ and sublinear at the infinity. □

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